Anisotropic heat transport in rigid solids

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1 The physical model

Let us consider a rigid heat conductor at rest, so that the referential and actual configurations coincide. The state space is spanned by the internal energy density e, the heat flux vector $\mathbf{q}$, together with their first order referential gradients ∇e and $\nabla \mathbf{q}$. Latin capital indices will denote quantities defined on the reference configuration, repetition implying summation. A superposed dot indicates the standard time derivative while the symbol $f_{,K}$ denotes partial derivation with respect to the coordinates X_K.

In the absence of heat supply the first law of Thermodynamics reads

$$\dot{e} + q_{J,J} = 0, \tag{1}$$

while the balance equation for the heat flux is

$$\dot{q}_I + P_{IJ,J} = r_I, \tag{2}$$

with P_{IJ} as the components of the flux of the heat flux and r_I as the components of the production of the heat flux.

In Extended Irreversible Thermodynamics the system above is closed by assigning the constitutive equations (closure relations) for the flux $\boldsymbol{P}$ and for the production $\boldsymbol{r}$ [1].

These equations must be postulated in such a way that the unilateral differential constraint

$$\dot{s} + J_{K,K} \geq 0, \tag{3}$$

where s denotes the entropy density and J_K are the components of the entropy flux, is satisfied. The functions s and $\boldsymbol{J}$ are constitutive quantities, too [1]. Restrictions are derived by imposing that the inequality (3) is satisfied for arbitrary thermodynamic processes, and result in a set of differential relationships involving the partial derivatives of $\boldsymbol{P}$, $\boldsymbol{r}$, s and $\boldsymbol{J}$ with respect to the thermodynamic variables.

Here we are looking for a suitable form of the constitutive equations which is simple but capable to represent anisotropy, nonlinearity and nonlocality. Hence, let's specialize them as follows

$$\begin{aligned} P_{IJ} &= A\delta_{IJ} + Bq_{K,K}\delta_{IJ} + 2L_{IJHK}\left\langle q_{H,K}\right\rangle + \\ &+ N_{IJHK}e_{,H}e_{,K} + 2M_{IJHK}\left\langle q_H e_{,K}\right\rangle, \end{aligned} \tag{4}$$

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$$
\begin{aligned}
r_I &= L_{IJ}q_J + M_{IJ}e_{,J} + N_{IJK}\left\langle e_{,J}q_K\right\rangle \\
&+ P_{IJK}\left\langle q_{J,K}\right\rangle + S_{IJK}e_{,J}e_{,K},
\end{aligned} \tag{5}
$$

$$
s = s_E(e) - \frac{1}{2}A_{IJ}q_Iq_J - \frac{1}{2}B_{IJ}e_{,I}e_{,J} - \frac{1}{2}D_{IJKL}q_{I,J}q_{K,L}, \tag{6}
$$

$$
\begin{aligned}
J_H &= \frac{\partial s}{\partial e}q_H + G_{HI}q_Jq_{I,J} + Z_{HI}e_{,J}q_{I,J} + \\
&+ M_{HI}q_Iq_{J,J} + N_{HI}e_{,I}q_{J,J}.
\end{aligned} \tag{7}
$$

In the equations above, all the tensorial coefficients are supposed to depend on the variables $\{e, q_K\}$ only. Moreover, in order to satisfy the principle of maximum entropy at the equilibrium, the tensors A_{IJ}, B_{IJ} and D_{IJKL} entering Eq. (6), are supposed to be positive semidefinite. Finally, here and in the following, the symbol $\langle f_{ABC..}\rangle_{(AB)}$ denotes the symmetrized tensor with respect to the indices A and B while the symbol $\langle f_{ABC..}\rangle$ denotes the symmetrized tensor with respect to all its indices.

2 Constitutive Theory

2.1 Restrictions placed by the principle of material frame indifference

Now we investigate the restrictions imposed on the model by the principle of material frame indifference [2]. Thus, let us consider a rigid transformation of coordinates

$$
\mathbf{X}' = \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t), \tag{8}
$$

where $\mathbf{Q}$ is a time-dependent proper orthogonal matrix and $\mathbf{c}$ is a regular vector function, and let us impose that the constitutive functions are invariant under the transformation (8). Such a requirement is equivalent to impose that scalar (f), vectorial ($\boldsymbol{v}$) and second-order tensorial ($\mathbf{T}$) constitutive quantities transform as follows [2]

$$
f' = f, \quad v'_I = Q_{IJ}v_J, \quad T'_{IJ} = Q_{IK}T_{KL}Q^T{}_{LJ}. \tag{9}
$$

In Eqs. (9), a superposed T denotes the transpose matrix.

As far as the elements of the state space are concerned, the following relationships are straightforward

$$
e' = e, \quad e'_{,I} = Q_{IJ}e_{,J}, \quad q'_I = Q_{IJ}q_J, \quad q'_{I,J} = Q_{IK}q_{K,L}Q^T_{L,J}. \tag{10}
$$

For the constitutive functions, instead, we require that

$$
P'_{IJ} = Q_{IL}P_{LM}Q^T_{MJ}, \tag{11}
$$

$$
r'_I = Q_{IJ}r_J, \tag{12}
$$

$$
s' = s, \tag{13}
$$

$$
J'_H = Q_{HK}J_K. \tag{14}
$$

We exploit the principle by applying the following methodology. After having calculated separately both the left-hand side and the right-hand side of Eqs. (11)-(14), by a direct comparison we infer the necessary and sufficient conditions which have to be valid in order to satisfy the material frame indifference.

As far as the flux of heat flux is concerned, the left-hand side of (11) reads

$$\begin{aligned} P'_{IJ} &= A'\delta_{IJ} + B'q'_{K,K}\delta_{IJ} + 2L'_{IJHK}\left\langle q'_{H,K}\right\rangle + \\ &+ N'_{IJHK}e'_{,H}e'_{,K} + 2M'_{IJHK}\left\langle q'_{H}e'_{,K}\right\rangle, \end{aligned} \tag{15}$$

with

$$\begin{aligned} L'_{IJHK} &= L_{IJHK}(e', q'_{K}), \\ N'_{IJHK} &= N_{IJHK}(e', q'_{K}), \\ M'_{IJHK} &= M_{IJHK}(e', q'_{K}). \end{aligned} \tag{16}$$

Due to the the relationship

$$q'_{I}{}^{2} = Q_{IJ}q_J Q_{IK}q_K = Q^T_{JI}Q_{IK}q_J q_K = \delta_{JK}q_J q_K = q_I q_I = q_I{}^2, \tag{17}$$

it follows that the functions which only depend on the energy density and on the squared heat flux are frame-invariant, so that in (15) we have

$$\begin{aligned} A' &= A, \\ B' &= B. \end{aligned} \tag{18}$$

On the other hand, the right-hand side of (11) reads

$$\begin{aligned} Q_{IL}P_{LM}Q^T_{MJ} &= AQ_{IL}\delta_{LM}Q^T_{MJ} + Bq_{K,K}Q_{IL}\delta_{LM}Q^T_{MJ} + 2Q_{IL}L_{LMHK}\left\langle q_{H,K}\right\rangle Q^T_{MJ} + \\ &+ Q_{IL}N_{LMHK}e_{,H}e_{,K}Q^T_{MJ} + 2Q_{IL}M_{LMHK}\left\langle q_H e_{,K}\right\rangle Q^T_{MJ}. \end{aligned} \tag{19}$$

Then, comparing Eq.s (15) and (19), we infer that Eq. (11) is satisfied if, and only if, the following relations are true

$$\langle L'_{IJHK}\rangle_{(HK)} = Q^T_{MH}Q_{IP}\langle L_{PSML}\rangle_{(ML)}Q^T_{JS}Q_{LK}, \tag{20}$$

$$\langle M'_{IJHK}\rangle_{(HK)} = Q^T_{HM}Q_{IP}\langle M_{PSML}\rangle_{(ML)}Q^T_{SJ}Q_{LK}, \tag{21}$$

$$Q_{IK}N_{KMSL}Q^T_{MJ} = N'_{IJHK}Q_{HS}Q_{LK}. \tag{22}$$

Let us consider now the production of heat flux. The left-hand side of (12) can be put in the form

$$\begin{aligned} r'_{I} &= L'_{IJ}q'_{J} + M'_{IJ}e'_{,J} + N'_{IJK}\left\langle e'_{,J}q'_{K}\right\rangle \\ &+ P'_{IJK}\left\langle q'_{J,K}\right\rangle + S'_{IJK}e'_{,J}e'_{,K}, \end{aligned} \tag{23}$$

with

$$\begin{aligned} L'_{IJ} &= L_{IJ}(e', q'_{K}), \\ M'_{IJ} &= M_{IJ}(e', q'_{K}), \\ N'_{IJK} &= N_{IJK}(e', q'_{K}), \\ P'_{IJK} &= P_{IJK}(e', q'_{K}), \\ S'_{IJK} &= S_{IJK}(e', q'_{K}). \end{aligned} \tag{24}$$

The right-hand side of (12), instead, reads

$$\begin{aligned} Q_{IJ} r_J &= Q_{IJ} L_{JK} q_K + Q_{IJ} M_{JK} e_{,K} + Q_{IJ} N_{JKM} \left\langle e_{,K} q_M \right\rangle \\ &+ Q_{IJ} P_{JKM} \left\langle q_{K,M} \right\rangle + Q_{IJ} S_{JKM} e_{,K} e_{,M}. \end{aligned} \tag{25}$$

Thus, by direct comparison, we infer that Eqs. (23) and (25) coincide if, and only if,

$$L'_{IJ} = Q_{IL} L_{LK} Q^T_{KJ}, \tag{26}$$
$$M'_{IJ} = Q_{IL} M_{LK} Q^T_{KJ}, \tag{27}$$

$$N'_{IJK} (Q_{JT} Q_{KM} + Q_{KT} Q_{JM}) = 2 Q_{IJ} \langle N_{JTM} \rangle_{(TM)}, \tag{28}$$

$$P'_{IJK} (Q_{JT} Q^T_{MK} + Q_{KT} Q^T_{MJ}) = 2 Q_{IJ} \langle P_{JTM} \rangle_{(TM)}, \tag{29}$$

$$S'_{IJM} Q_{JT} Q_{MK} = Q_{IJ} S_{JKT}. \tag{30}$$

Let us investigate now the restrictions placed by the objectivity principle on the specific entropy. The left-hand side of (13) reads

$$s' = s'_E(e') - \frac{1}{2} A'_{IJ} q'_I q'_J - \frac{1}{2} B'_{IJ} e'_{,I} e'_{,J} - \frac{1}{2} D'_{IJKL} q'_{I,J} q'_{K,L}, \tag{31}$$

with

$$\begin{aligned} A'_{IJ} &= A_{IJ}(e', q'_K), \\ B'_{IJ} &= B_{IJ}(e', q'_K), \\ D'_{IJKL} &= D_{IJKL}(e', q'_K). \end{aligned} \tag{32}$$

Then, by taking into account the relationships (11)-(14), it turns out that Eqs. (31) and (6) coincide if, and only if,

$$\begin{aligned} A'_{IJ} &= Q_{IK} A_{KM} Q^T_{MJ}, \\ B'_{IJ} &= Q_{IK} B_{KM} Q^T_{MJ}, \\ Q^T_{MI} D'_{IJKL} Q_{LT} &= Q_{JN} D_{MNST} Q^T_{SK}. \end{aligned} \tag{33}$$

Finally, let us consider the entropy flux. The left-hand side of (14) reads

$$\begin{aligned} J'_H &= \left(\frac{\partial s}{\partial e}\right)' Q_{HK} q_K + G'_{HI} Q_{JM} q_M Q_{IK} q_{K,L} Q^T_{LJ} + Z'_{HI} Q_{JM} e_{,M} Q_{IK} q_{K,L} Q^T_{LJ} + \\ &+ M'_{HI} Q_{IK} q_K Q_{JM} q_{M,N} Q^T_{NJ} + N'_{HI} Q_{IK} e_{,K} Q_{JM} q_{M,N} Q^T_{NJ}, \end{aligned} \tag{34}$$

with

$$\begin{aligned} G'_{HI} &= G_{HI}(e, q_K{}'), \\ Z'_{HI} &= F_{HI}(e, q_K{}'), \\ M'_{HI} &= M_{HI}(e, q_K{}'), \\ N'_{HI} &= N_{HI}(e, q_K{}'). \end{aligned} \tag{35}$$

Of course,

$$\Big(\frac{\partial s}{\partial e}\Big)' = \Big(\frac{\partial s}{\partial e}\Big). \tag{36}$$

On the other hand, the right-hand side of (14) can be expressed as follows

$$\begin{aligned} Q_{HK} J_K &= \Big(\frac{\partial s}{\partial e}\Big) Q_{HK} q_K + Q_{HK} G_{KI} q_J q_{I,J} + Q_{HK} Z_{KI} e_{,J} q_{I,J} + \\ &+ Q_{HK} M_{KI} q_I q_{J,J} + Q_{HK} N_{KI} e_{,I} q_{J,J}. \end{aligned} \tag{37}$$

By the relations (14), (34) and (37) we get the conditions

$$G'_{HI} = Q_{HS} G_{SK} Q^T_{KI}, \tag{38}$$
$$Z'_{HI} = Q_{HS} Z_{SK} Q^T_{KI}, \tag{39}$$
$$M'_{HI} = Q_{HS} M_{SK} Q^T_{KI}, \tag{40}$$
$$N'_{HI} = Q_{HS} N_{SK} Q^T_{KI}. \tag{41}$$

2.2 Restrictions placed by second law of Thermodynamics

According to the generalized Coleman-Noll procedure [3], in exploiting the second law of Thermodynamics we take into account not only the governing equations for the unknown fields, but their gradient extension too, up to the order of the gradients entering the state space. To this end, we eliminate from the entropy inequality all the time derivatives of the thermodynamic variables by substituting the balance equations and their gradients. In our derivation it is necessary to distinguish between *higher derivatives* and *highest derivatives*. The higher derivatives are the spatial derivatives whose order is higher than that of the gradients entering the state space. The highest derivatives, instead, are both the time derivatives of the elements of the state space which cannot be expressed through the governing equations as functions of the thermodynamic variables, and the higher derivatives whose order is the highest one [4]. In the classical case, the highest derivatives coincide with the higher ones.

Taking into account the state space, we can write the gradients of the balances of energy and heat flux as follows

$$\dot{e}_{,I} + q_{K,KI} = 0, \tag{42}$$

$$\dot{q_{I,J}} + \frac{\partial^2 P_{IK}}{\partial e^2} e_{,K} e_{,J} + \frac{\partial^2 P_{IK}}{\partial e \partial q_L} e_{,K} q_{L,J} + \frac{\partial^2 P_{IK}}{\partial e \partial e_{,M}} e_{,K} e_{M,J} + \frac{\partial^2 P_{IK}}{\partial e \partial q_{L,M}} e_{,K} q_{L,MJ} + \frac{\partial P_{IK}}{\partial e} e_{,KJ}$$

$$+ \frac{\partial^2 P_{IK}}{\partial e \partial e_{,L}} e_{,LK} e_{,J} + \frac{\partial^2 P_{IK}}{\partial e_{,L} \partial e_{,M}} e_{,LK} e_{,MJ} ++ \frac{\partial^2 P_{IK}}{\partial e_{,L} \partial q_L} e_{,LK} q_{,LJ} ++ \frac{\partial^2 P_{IK}}{\partial e_{,L} \partial q_{L,M}} e_{,LK} q_{L,MJ} + \frac{\partial P_{IK}}{\partial e_{,L}} e_{,LKJ}$$

$$+ \frac{\partial^2 P_{IK}}{\partial q_L \partial e} q_{L,K} e_{,J} + \frac{\partial^2 P_{IK}}{\partial q_L \partial e_{,M}} q_{L,K} e_{,MJ} + \frac{\partial^2 P_{IK}}{\partial q_L \partial q_M} q_{L,K} q_{M,J} + \frac{\partial^2 P_{IK}}{\partial q_L \partial q_{N,M}} q_{L,K} q_{N,MJ} + \frac{\partial P_{IK}}{\partial q_L} q_{L,KJ}$$

$$+ \frac{\partial^2 P_{IK}}{\partial q_{L,M} \partial e} q_{L,MK} e_{,J} + \frac{\partial^2 P_{IK}}{\partial q_{L,M} \partial e_{,N}} q_{L,MK} e_{,NJ} + \frac{\partial^2 P_{IK}}{\partial q_{L,M} \partial q_N} q_{L,MK} q_{N,J} + \frac{\partial^2 P_{IK}}{\partial q_{L,M} \partial q_{N,P}} q_{L,MK} q_{N,PJ}$$

$$+\frac{\partial P_{IK}}{\partial q_{L,M}} q_{L,MKJ} = \frac{\partial r_I}{\partial e} e_{,J} + \frac{\partial r_I}{\partial e_{,L}} e_{,LJ} + \frac{\partial r_I}{\partial q_L} q_{L,J} + \frac{\partial r_I}{\partial q_{L,M}} q_{L,MJ}. \tag{43}$$

Moreover, the entropy inequality on the state space reads

$$\frac{\partial s}{\partial e}\dot{e} + \frac{\partial s}{\partial e_{,L}}\dot{e}_{,L} + \frac{\partial s}{\partial q_L}\dot{q}_L + \frac{\partial s}{\partial q_{L,M}}\dot{q}_{L,M} + \frac{\partial J_J}{\partial e} e_{,J} + \frac{\partial J_J}{\partial e_{,L}} e_{,LJ} + \frac{\partial J_J}{\partial q_L} q_{L,J} + \frac{\partial J_J}{\partial q_{L,M}} q_{L,MJ} \geq 0. \tag{44}$$

The further step will consist in eliminating the time derivatives from Eq. (44) by substitution of Eqs. (1), (2), (42) and (43). In this way, one obtains a generalized entropy inequality in which the highest derivatives are the tensors $e_{,MKJ}$ and $q_{L,MKJ}$, while the higher derivatives are the tensors $e_{,MK}$ and $q_{L,MK}$. Restrictions on the constitutive functions are obtained by putting equal to zero the terms which are linear either in the highest or in the higher derivatives [4]. It turns out that the constitutive quantities must obey the following set of differential constraints [4]:

$$\Big\langle D_{IJNP} q_{N,P} \Big(L_{IKLM} + L_{IKML} \Big) + D_{KJNP} q_{N,P} B \delta_{LM} \Big\rangle_{(KLM)} = 0, \tag{45}$$

$$\Big\langle D_{IJSP} q_{S,P} \Big(N_{IKLH} e_{,H} + N_{IKML} e_{,M} + M_{IKNL} q_N + M_{IKLH} q_H \Big) \Big\rangle = 0, \tag{46}$$

$$\begin{aligned}
&\Big\langle G_{KL} q_N + Z_{KL} e_{,N} + M_{KH} q_H \delta_{LN} + N_{KH} e_{,H} \delta_{LN} + D_{IJHM} q_{H,M} \frac{\partial}{\partial q_P} \Big(B \delta_{LN} \delta_{IK} + 2L_{IK<LN>} \Big) q_{P,J} \\
&+ D_{IJHM} q_{H,M} \frac{\partial}{\partial e} \Big(B \delta_{LN} \delta_{IK} + 2L_{IK<LN>} \Big) e_{,J} + D_{IKHM} q_{H,M} \frac{\partial}{\partial q_L} \Big(A \delta_{IN} + B q_{M,M} \delta_{IN} \\
&+ 2L_{INHK} \left\langle q_{H,K} \right\rangle + N_{INHK} e_{,H} e_{,K} + 2M_{INHK} \left\langle q_H e_{,K} \right\rangle \Big) + D_{IKHM} q_{H,M} \frac{\partial}{\partial q_S} \Big(B \delta_{LN} \delta_{IP} \\
&+ 2L_{IP<LN>} \Big) q_{S,P} + D_{IKHM} q_{H,M} \frac{\partial}{\partial e} \Big(B \delta_{LN} \delta_{IJ} + 2L_{IJ<LN>} \Big) e_{,J} + B_{IK} e_{,I} \delta_{LN} - 2D_{IKHM} q_{H,M} P_{I<LN>} \\
&+ \Big(A_{IJ} q_J + \frac{1}{2} \frac{\partial B_{NZ}}{\partial q_I} e_{,N} e_{,Z} + \frac{1}{2} \frac{\partial D_{NJKL}}{\partial q_I} q_{N,J} q_{K,L} \Big) \Big(B \delta_{LN} \delta_{IK} + 2L_{IK<LN>} \Big) \Big\rangle_{(LN)} = 0,
\end{aligned} \tag{47}$$

$$\begin{aligned}
&\Big\langle Z_{HL} q_{L,M} + N_{HM} q_{L,L} + 2D_{IHNS} q_{N,S} \Big(\frac{\partial N_{IKHM}}{\partial q_L} e_{,H} + M_{IK<LM>} \Big) q_{L,K} + 2D_{IPKS} q_{K,S} \Big(\frac{\partial N_{IHLM}}{\partial q_N} e_{,L} \\
&+ M_{IH<NM>} \Big) q_{N,P} + 2D_{IPKS} q_{K,S} \frac{\partial}{\partial e} \Big(N_{IHLM} e_{,L} + N_{IMLN} e_{,L} \Big) e_{,P} + D_{IHKS} q_{K,S} \frac{\partial}{\partial e} \Big(A \delta_{IM} + B q_{L,L} \delta_{IM} \\
&+ 2L_{IMLN} \left\langle q_{L,N} \right\rangle + 2N_{IMLN} e_{,L} e_{,N} + 2M_{IMLN} \left\langle q_L e_{,N} \right\rangle \Big) + 2D_{IHPS} q_{P,S} \Big(N_{IKLM} e_{,L} + M_{IK<LM>} q_L \Big) e_{,K} \\
&+ 2A_{IJ} q_J \Big(N_{IHLM} e_{,L} + M_{IH<LM>} q_L \Big) - D_{IHKS} q_{K,S} \Big(M_{IM} + N_{I<LM>} q_L + 2S_{IJM} e_{,J} \Big) \Big\rangle = 0.
\end{aligned} \tag{48}$$

In order to get more manageable expressions, let us neglect the terms with the gradient of the internal energy in $\boldsymbol{P}$, namely, let us pursue our analysis under the hypothesis

$$M_{IHKL} = N_{IHKL} = 0. \tag{49}$$

Thus, the set of thermodynamic restrictions above reduces to

$$\Big\langle D_{IJNP}q_{N,P}\Big(L_{IKLM}+L_{IKML}\Big)+D_{KJNP}q_{N,P}B\delta_{LM}\Big\rangle_{(KLM)}=0, \tag{50}$$

$$\begin{aligned}&\Big\langle G_{KL}q_N+Z_{KL}e_{,N}+M_{KH}q_H\delta_{LN}+N_{KH}e_{,H}\delta_{LN}+D_{IJHM}q_{H,M}\frac{\partial}{\partial q_P}\Big(B\delta_{LN}\delta_{IK}+2L_{IK<LN>}\Big)q_{P,J}\\&+D_{IJHM}q_{H,M}\frac{\partial}{\partial e}\Big(B\delta_{LN}\delta_{IK}+2L_{IK<LN>}\Big)e_{,J}+D_{IKHM}q_{H,M}\frac{\partial}{\partial q_L}\Big(A\delta_{IN}+Bq_{M,M}\delta_{IN}\\&+2L_{INHK}\left\langle q_{H,K}\right\rangle\Big)+D_{IKHM}q_{H,M}\frac{\partial}{\partial q_S}\Big(B\delta_{LN}\delta_{IP}+2L_{IP<LN>}\Big)q_{S,P}\\&+D_{IKHM}q_{H,M}\frac{\partial}{\partial e}\Big(B\delta_{LN}\delta_{IJ}+2L_{IJ<LN>}\Big)e_{,J}-2D_{IKHM}q_{H,M}P_{I<LN>}+B_{IK}e_{,I}\delta_{LN}\\&+\Big(A_{IJ}q_J+\frac{1}{2}\frac{\partial B_{NZ}}{\partial q_I}e_{,N}e_{,Z}+\frac{1}{2}\frac{\partial D_{NJKL}}{\partial q_I}q_{N,J}q_{K,L}\Big)\Big(B\delta_{LN}\delta_{IK}+2L_{IK<LN>}\Big)\Big\rangle_{(LN)}=0,\end{aligned} \tag{51}$$

$$\begin{aligned}&\Big\langle Z_{HL}q_{L,M}+N_{HM}q_{L,L}+D_{IHKS}q_{K,S}\frac{\partial}{\partial e}(A\delta_{IM}+Bq_{L,L}\delta_{IM}+2L_{IMLN}\left\langle q_{L,N}\right\rangle)\\&-D_{IHKS}q_{K,S}\Big(M_{IM}+N_{I<ML>}q_L+2S_{IJM}e_{,J}\Big)\Big\rangle=0.\end{aligned} \tag{52}$$

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